

MAS 204 CALCULUS III

Monday 19th May 1997, 2.30 p.m.

Duration: 2 hours

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators may be used in this examination but any programming, graph plotting or algebraic facility may not be used. Please state on your answer book the name and type of machine used.

You are reminded of the following, which you may use without proof.

In orthogonal curvilinear coordinates (u_1, u_2, u_3) , with corresponding unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and arc-length parameters h_1, h_2, h_3 , the divergence of a vector field $\mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3$ is given by

$$\nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} (h_2 h_3 F_1) + \frac{\partial}{\partial u_2} (h_3 h_1 F_2) + \frac{\partial}{\partial u_3} (h_1 h_2 F_3) \right],$$

and the curl of the same vector field is given by

$$\nabla \times \mathbf{F} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \mathbf{e}_1 & h_2 \mathbf{e}_2 & h_3 \mathbf{e}_3 \\ \partial/\partial u_1 & \partial/\partial u_2 & \partial/\partial u_3 \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix};$$

also in such a coordinate system, for scalar field f ,

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial f}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial f}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial f}{\partial u_3} \right) \right].$$

In spherical polar coordinates $(u_1, u_2, u_3) \equiv (r, \theta, \phi)$, the arc-length parameters are $h_1 = 1, h_2 = r, h_3 = r \sin \theta$.

In cylindrical polar coordinates $(u_1, u_2, u_3) \equiv (r, \theta, z)$, the arc-length parameters are $h_1 = 1, h_2 = r, h_3 = 1$.

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SECTION A Each question carries 8 marks. You should attempt ALL questions.

A1. Evaluate

$$(a) \nabla f, \quad (b) \nabla \cdot \mathbf{F}, \quad (c) \nabla \times \mathbf{F}, \quad (d) \nabla^2 f,$$

where $f = x^2 + y^3 + z^4$ and $\mathbf{F} = xy^2z^3\mathbf{i} + x^3y^2z\mathbf{k}$.

A2. Evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F} = 6z\mathbf{i} + y^2\mathbf{j} + 12x\mathbf{k}$$

and C is a path going from $(0, 1, 0)$ to $(0, 1, \pi/3)$ and is described in parametric form as

$$C: \mathbf{r} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + (t/6)\mathbf{k}, \quad 0 \leq t \leq 2\pi.$$

A3. Using the index notation or otherwise, prove the identities (a) $\nabla(\mathbf{a} \cdot \mathbf{r}) = \mathbf{a}$ and (b) $\nabla \cdot (\mathbf{a} \times \mathbf{r}) = 0$, where $\mathbf{r}$ is the position vector and $\mathbf{a}$ is a constant vector.

A4. State the Divergence Theorem. Using the Divergence Theorem or otherwise, evaluate $\int_S \mathbf{F} \cdot \mathbf{n} dS$ where $\mathbf{F} = xy^2z\mathbf{e}_z$ and S is the surface of the unit cube, $\mathbf{n}$ being the outward normal.

[The unit cube is the cube whose vertices are at $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 0, 1)$, $(1, 1, 1)$, $(0, 1, 1)$.]

A5. Write down Laplace's equation in cylindrical polar coordinates (r, θ, z) . Show that the following are both solutions of Laplace's equation:

$$(a) f = r^2 \cos 2\theta, \quad (b) f = r^2 z \sin 2\theta.$$

A6. Sketch a few representative streamlines for the following fluid velocity fields:

$$(a) \mathbf{u} = x\mathbf{i} + y\mathbf{j}, \quad (b) \mathbf{u} = x\mathbf{i} - y\mathbf{j}.$$

A7. Starting from Maxwell's equations,

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \rho/\epsilon_0, & \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, & \nabla \times \mathbf{B} &= \mu_0 \mathbf{j} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \end{aligned}$$

(where $c^2 = 1/(\epsilon_0 \mu_0)$, ϵ_0 and μ_0 are constants, and t is time), derive the equation

$$\partial \rho / \partial t + \nabla \cdot \mathbf{j} = 0.$$

[Although you do not need to use the fact, the symbols have their usual physical interpretation: $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields, $\mathbf{j}$ is the current density and ρ is the charge density.]

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SECTION B Each question carries 25 marks. You may attempt all questions but only marks for the best 2 questions will be counted.

- B1. Volume V is closed and singly-connected, with surface S . Prove that $w = 0$ is the only solution of

$$\begin{cases} \nabla^2 w = 0 & \text{in } V \\ w = 0 & \text{on } S. \end{cases}$$

[Hint: first show that $\int_S w \nabla w \cdot \mathbf{n} \, dS = \int_V w \nabla^2 w + (\nabla w)^2 \, dV$.]

Hence or otherwise, show that there is at most one function u that satisfies

$$\begin{cases} \nabla^2 u = f(\mathbf{r}) & \text{in } V \\ u = g(\mathbf{r}) & \text{on } S \end{cases}$$

for given functions f and g .

Let S be the sphere $r = a$, where r is the usual spherical polar coordinate and a is a positive constant, and let V be the volume enclosed by S . Find all solutions of the problem

$$\begin{cases} \nabla^2 u = r & \text{in } V \\ u = a^3 & \text{on } S. \end{cases}$$

- B2. Using the method of Frobenius, find two independent series solutions of the second-order differential equation

$$4x^2 \frac{d^2 y}{dx^2} + \left(9x^3 + \frac{7}{16} \right) y = 0.$$

You should exhibit the first few terms of each solution, and consider the convergence of the solutions.

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B3. Vector fields $\mathbf{A}$, $\mathbf{B}$ satisfy

$$\mathbf{B} = \nabla \times \mathbf{A} .$$

Prove that $\nabla \cdot \mathbf{B} = 0$. Now suppose that a new vector field $\mathbf{A}'$ is defined by $\mathbf{A}' = \mathbf{A} + \nabla \phi$, where ϕ is a given (but arbitrary) scalar field. Prove that $\nabla \times \mathbf{A}' = \mathbf{B}$.

Vector field $\mathbf{B}$ satisfies $\nabla \cdot \mathbf{B} = 0$ everywhere. $\mathbf{B}$ also satisfies

$$\nabla \times \mathbf{B} = f \mathbf{B}$$

where $f(\mathbf{r})$ is some function of position. Prove that $(\nabla \times \mathbf{B}) \times \mathbf{B} = 0$ and prove also that $\mathbf{B}$ and f must satisfy $\mathbf{B} \cdot \nabla f = 0$.

Show that the field

$$\mathbf{B} = B_0 \left[(\sin kz) \mathbf{i} + (\cos kz) \mathbf{j} \right] ,$$

where B_0 and k are constants, satisfies $\nabla \cdot \mathbf{B} = 0$ and $(\nabla \times \mathbf{B}) \times \mathbf{B} = 0$. Find an example of a vector field $\mathbf{A}$ such that $\nabla \times \mathbf{A} = \mathbf{B}$ for this particular $\mathbf{B}$.

B4. Let u and v be curvilinear coordinates (not necessarily orthogonal) in the (x, y) plane. If

$$d\mathbf{x} = h_1 \mathbf{e}_1 du + h_2 \mathbf{e}_2 dv ,$$

$\mathbf{e}_1$ and $\mathbf{e}_2$ being unit vectors in the directions of u and v increasing respectively, if ϕ is a scalar function of u and v , and if

$$\nabla \phi = f(u, v) \mathbf{e}_1 + g(u, v) \mathbf{e}_2 ,$$

find expressions for $f(u, v)$ and $g(u, v)$.

In the case $x = u$, $y = v/u$, show that

$$f = \left(1 + \frac{v^2}{u^4} \right)^{1/2} \left(\frac{\partial \phi}{\partial u} + \frac{v}{u} \frac{\partial \phi}{\partial v} \right) .$$

[End of paper]